§2.5 Chain rule.

Key formula: $[f(g(x))]' = f'[g(x)] \cdot g'(x)$. Derivative of composition functions.

In Leibnitz notation: $\frac{df}{d\cancel{x}} = \frac{df}{dg} \cdot \frac{dg}{dx}$.

Rmk1: $f(g(x))$ is called the composition of f following g. $f(x)$: Outer function, $g(x)$: Inner function.

e.g.1 $y=\sqrt{x^2+1}$ can be re-written as the composition of $f(x)=\sqrt{x}$ following $g(x)=x^2+1$

as $y=\sqrt{x^2+1} = f(g(x))$. Outer function $\sqrt{\blacksquare}$; Inner function: $\blacksquare^2+1$.

Rmk2: $f'[g(x)]$ means: find $f'(x)$ first, then plug in $g(x)$.

eg.2 Find y' of $y=\sqrt{x^2+1}$.

Solution: Step 1: $f(x)=\sqrt{x}$, $g(x)=x^2+1$

Step 2.1: $f'(x) = (x^{\frac{1}{2}})' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} \cdot x^{-\frac{1}{2}}$

Step 2.2: Plug in $g(x)$, $f'[g(x)] = \boxed{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}}$

Step 3: $g'(x) = (x^2+1)' = 2x+0 = \boxed{2x}$

Step 4: $y' = (\sqrt{x^2+1})' = f'[g(x)] \cdot g'(x) = \boxed{\frac{1}{2} \cdot (x^2+1)^{-\frac{1}{2}} \cdot 2x} = (x^2+1)^{-\frac{1}{2}} \cdot x$.

eg.3. Let $f(t) = \frac{\cos(2\sin t)}{3}$. Find $f'(t)$.

Solution: $f(t) = \underbrace{\frac{1}{3}\cos}_{\text{outer}}[\underbrace{2\sin t}_{\text{inner}}]$

Derivative of outer: $\left(\frac{1}{3}\cos \blacksquare\right)' = \frac{1}{3}[\cos \blacksquare]' = \frac{1}{3}\cdot(-\sin \blacksquare) = -\frac{1}{3}\sin \blacksquare$

Plug in inner: $-\frac{1}{3}\sin(2\sin t)$

Derivative of inner: $(2\sin t)' = 2(\sin t)' = 2\cos t$.

Chain rule: $f'(t) = \left[\frac{1}{3}\cos(2\sin t)\right]' = \boxed{-\frac{1}{3}\sin(2\sin t) \cdot 2\cos t} = -\frac{2}{3}\sin(2\sin t)\cdot \cos t$.

eg 4. Let $y = \underbrace{\tan}_{\text{outer}}\underbrace{(3x)}_{\text{inner}}$ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$\left.\begin{aligned} &(\tan \blacksquare)' = \sec^2(\blacksquare) \xrightarrow[3x]{\text{Plug in inner}} \sec^2(3x) \\ &(3x)' = 3 \end{aligned}\right\} \Rightarrow \frac{dy}{dx} = (\tan(3x))' = \boxed{\sec^2(3x)\cdot 3}$$

$\frac{dy}{dx} = y' = 3\cdot\sec^2(3x)$. $\quad \frac{d^2y}{dx^2} = y'' = (y')' = (3\sec^2(3x))' = \boxed{18\cdot\sec^2(3x)\cdot\tan(3x)}$

$= 3[\sec(3x)]^2$ Outer: $3[\blacksquare]^2$. Inner: $\sec(3x)$.

(1st chain):

$$\left.\begin{aligned} &(\text{outer})' = (3\blacksquare^2)' = 3\cdot 2\cdot\blacksquare \xrightarrow[\sec(3x)]{\text{Plug in Inner}} 6\cdot\sec(3x) \\ &(\text{inner})' = (\underbrace{\sec}_{\text{outer}}\underbrace{(3x)}_{\text{inner}})' = [\tan(3x)\cdot\sec(3x)]\cdot 3 \end{aligned}\right\} \Rightarrow (3\sec^2(3x))' = [6\sec(3x)][\tan(3x)\sec(3x)\cdot 3]$$

$= 18\cdot\sec^2(3x)\cdot\tan(3x)$.

(2nd chain):

$$\left.\begin{aligned} &(\sec\bullet)' = \tan\bullet\cdot\sec\bullet \xrightarrow{3x} \tan(3x)\sec(3x) \\ &(3x)' = 3. \end{aligned}\right\}$$

Remark: It is actually the composition of three functions, which requires chain rule TWICE.

$$(f[g(h(x))])' = f'(g(h(x)))\cdot[g(h(x))]' = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x).$$

eg.5. (Product rule + Chain rule). Find the derivative of $f(x) = \frac{x}{\tan(x^2-1)}$.

(S16). $f'(x) = \dfrac{x'\cdot\tan(x^2-1) - x\cdot[\tan(x^2-1)]'}{[\tan(x^2-1)]^2} = \boxed{\dfrac{\tan(x^2-1) - x\cdot[\sec^2(x^2-1)\cdot 2x]}{[\tan(x^2-1)]^2}}$

$[\underbrace{\tan}_{\text{outer}}\underbrace{(x^2-1)}_{\text{inner}}]' = \underbrace{[\sec^2(x^2-1)]}_{(\text{outer})'}\cdot\underbrace{(x^2-1)'}_{(\text{inner})'} = \sec^2(x^2-1)\cdot 2x$.

eg.6. (S16, MC) Let $h(x) = f(g(x))$ where $\boxed{g(1)=2}$, $\boxed{g'(1)=3}$, $f(1)=4$, $f'(1)=5$, $f(2)=6$, $\boxed{f'(2)=7}$

Find $h'(1)$. Rmk: Compute $h'(x)$ then plug in $x=1$.

$$h'(x) = [f(g(x))]' = f'(g(x))\cdot g'(x)$$

$$\Rightarrow h'(1) = f'(g(1))\cdot g'(1) = f'(2)\cdot 3 = 7\cdot 3 = \boxed{21}$$

§2.7. Rates of Change.

Key points: Functions of Motion.

We consider the following physical quantities describing motion as functions of time t.

- Position or displacement $s(t)$. (in feet). $O \longrightarrow s(t) \longrightarrow$
- Velocity at t: $v(t) = s'(t)$. (in ft/s) • $V_{ave} = \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$ (sec 1.4)
- Speed: magnitude of velocity, i.e., $|v|$. • Distance
- Acceleration: $a(t) = v'(t)$. (in ft/s^2)

Rmk: $v > 0 \Leftrightarrow$ moving forward $\Leftrightarrow$ s is increasing

$v < 0 \Leftrightarrow$ moving backward $\Leftrightarrow$ s is decreasing

$|v|$ is increasing $\Leftrightarrow$ speed up, $|v|$ is decreasing $\Leftrightarrow$ slow down.

$a > 0 \Leftrightarrow v$ is increasing $\begin{cases} v>0, & |v| \text{ is increasing} \\ v<0, & |v| \text{ is decreasing} \end{cases}$

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eg. 1. (§1.6) The position of a particle moving along the x-axis is $x(t) = t^4 - 4t^3 + 1$, $t > 0$

(a). when is the velocity negative? (b) when is the acceleration negative?

solution: $v(t) = (t^4 - 4t^3 + 1)' = 4 \cdot t^3 - 4 \cdot 3t^2 + 0 = \boxed{4t^3 - 12 \cdot t^2}$

$v(t) < 0 \Leftrightarrow 4t^3 - 12 \cdot t^2 < 0 \Leftrightarrow 4t^2(t-3) < 0$

The velocity is negative when $t < 3$.

$a(t) = v'(t) = (4t^3 - 12t^2)' = 4 \cdot 3t^2 - 12 \cdot 2t = 12t^2 - 24t$.

$a(t) < 0 \Leftrightarrow 12t^2 - 24t < 0$

$\Leftrightarrow 12t \cdot (t-2) < 0$

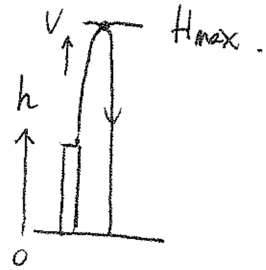
The acceleration is negative when $t < 2$.

Rmk: (a) is equivalent to ask "when is the particle moving in the negative direction" e.g. 2. A ball is thrown upward from the top of a building 50 feet tall.

The height of the ball is described by the function. $h(t) = -t^2 + 5t + 50$.

(a) When does the ball reach the maximum height?

(b). When does the ball reach the ground with what velocity?



solution: (a) Maximum height $\Leftrightarrow$ velocity zero.

$$v'(t) = (-t^2+5t+50)' = -2t+5 = 0$$

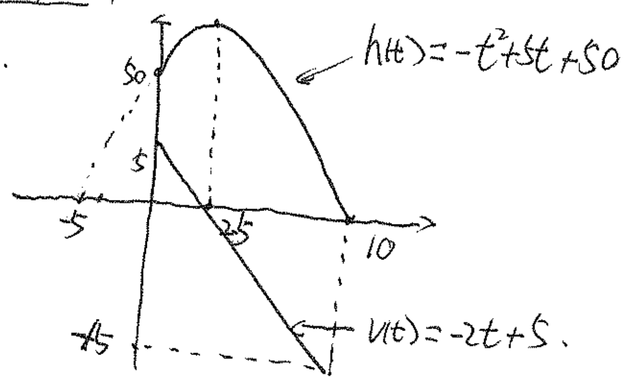
$$\Rightarrow \boxed{t = 2.5 \quad (s)}$$

(b). $h(t) = -t^2+5t+50 = 0 \Rightarrow t^2-5t-50=0 \Rightarrow (t+5)(t-10)=0$

$$\Rightarrow \boxed{t = 10 \quad (s)}$$

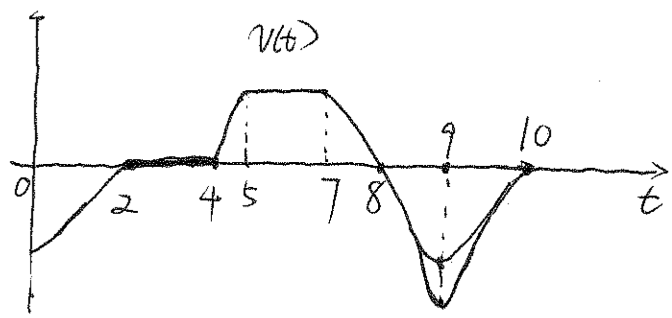
and $v(10) = -2\cdot 10 + 5 = \boxed{-15 \quad ft/s}$

(c). Sketch the graph of h and v.



e.g. 3. (WW #9)

Give the graph of $v(t)$ in $t \in [0, 10]$ as follow.



At $t=9$, the particle reaches its maximum speed.

The interval the particle moves forward: $v>0$ $\quad t \in (4,8)$

The interval the particle moves backward: $v<0$ $\quad t \in (0,2) \cup (8,10)$

The interval the particle stops: $v=0$ $\quad t \in [2,4]$

The interval the particle speed up: $|v|$ is increasing: $(4,5) \cup (8,9)$

The interval the particle slow down: $|v|$ is decreasing: $(0,2) \cup (7,8) \cup (9,10)$

The interval the acceleration is positive: v is increasing $(0,2) \cup (4,5) \cup (9,10)$

The interval the acceleration is negative: v is decreasing $(7,9)$

The interval the acceleration is zero: v is constant $(2,4) \cup (5,7)$

§2.6 Implicit Differentiation.

Key points: • Explicit/Implicit functions

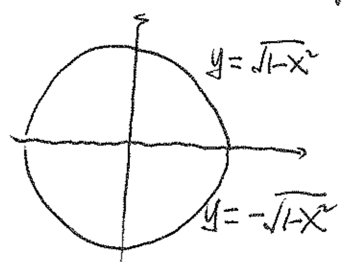
• Implicit Differentiation Rule.

Explicit function: From equation $4x+2y=6$, we can solve for y as.

$y=-2x+3$. EXPLICITLY.

Implicit function: While from $x^2+y^2=1$, it is not so convinient to solve for y.

Instead of solving y (as two functions), we assume an implicit relation $y=y(x)$ (unknown function) which satisfies the above equation. Such unknown functions are called IMPLICIT functions.



The goal is HOW TO TAKE THE DERIVATIVES of such unknow functions.

And the tangent line to the given curve (as an equation of x, y).

e.g. 1. Suppose y and x satisfy the implicit equation $\frac{x^2}{4}+\frac{y^2}{5}=1$

Find $y'=\frac{dy}{dx}$.

Solution: Step 1: Take derivatives (with respect to x) both sides of the equation:

$$\left(\frac{x^2}{4}+\frac{y^2}{5}\right)'=(1)'=0$$

$$\Leftrightarrow \left(\frac{x^2}{4}\right)'+\left(\frac{y^2}{5}\right)'=0 \Leftrightarrow \frac{1}{4}\cdot 2x+\frac{1}{5}\cdot(y^2)'=0. \quad (*)$$

Caution: $\boxed{(y^2)' \neq 2\cdot y.}$ $y=y(x)$ is a FUNCTION of x.

we need to apply CHAIN RULE. with outer function $[\bullet]^2$ and inner $y(x)$

$$\left([y(x)]^2\right)'=[2y(x)]\cdot y'(x) = 2y\cdot y'$$

i.e. $(*) \Leftrightarrow \frac{1}{4}\cdot 2x+\frac{1}{5}\cdot 2y\cdot y'=0 \Leftrightarrow [\frac{2}{5}y]\cdot y'=-\frac{1}{2}x$

Step 2: Solve for y' as a function of x, y: $\boxed{y'=-\frac{5}{4}\cdot\frac{x}{y}}$

e.g.2 Suppose x, y satisfy the implicit equation $y^2 + x\cdot y + x^3 = 3$.
(F16). Find $y' = \frac{dy}{dx}$ as a function of x, y.

Solution: Take derivatives both sides of the equation: $(y^2 + x\cdot y + x^3)' = 3'$

Notice $(y^2)' = 2y\cdot y'$, (chain rule). and $(x\cdot y)' = x'\cdot y + x\cdot y'$
$= 1\cdot y + x\cdot y'$

Therefore,

Caution: $y' \neq 1$ since $y = y(x)$ is a function of x

(☆) $2y\cdot y' + y + x\cdot y' + 3x^2 = 0.$

Then fix x, y (treat them as some numbers) and solve for y'.

$(2y + x)\cdot y' + y + 3x^2 = 0 \iff (2y+x)\cdot y' = -y - 3x^2$

$\iff \boxed{y' = \frac{-y-3x^2}{2y+x}}$

e.g.3. Consider the curve $x^2 + y^3 + x y = 1$.
(S16). (a). Find the slope of the tangent line of the curve at the point $(2,-1)$.
(b) Find the equation of the tangent line.

Rmk: Recall slope of the tangent line = derivative of the "function" evaluated at "this point".
Here "the function" is the implicit function $y = y(x)$ and the point (x-coordinate) is $x = 2$.
i.e. (a) is equivalent to find $\boxed{\frac{dy}{dx}\Big|_{x=2}}$

(a). Take derivative both sides: $(x^2 + y^3 + x\cdot y)' = (1)' \iff (x^2)' + (y^3)' + (xy)' = 0.$
$(x^2)' = 2x.$ $(x\cdot y)' = x'\cdot y + x\cdot y' = y + x\cdot y'$ (product rule).
$(y^3)'$: outer function: $\blacksquare^3$, $((\blacksquare)^3)' = 3\blacksquare^2 \xrightarrow[\text{inner } y(x)]{\text{Plug in}} 3[y(x)]^2$
inner function: $y(x)$, $y'(x)$

Chain rule gives us: $(y^3)' = 3y^2\cdot y'$. Therefore, $2x + 3y^2\cdot y' + y + x\cdot y' = 0$
Plug in $(2,-1)$, ie., $x=2$, $y=-1$. $2\cdot 2 + 3(-1)^2\cdot y' + (-1) + 2\cdot y' = 0$
$\Rightarrow 4 + 3\cdot y' - 1 + 2\cdot y' \Rightarrow 5y' = -3 \Rightarrow \boxed{y' = -\frac{3}{5}}$

(b) Point slope formula: $\underset{\text{point}}{(2,-1)}$; $\underset{\text{slope}}{-\frac{3}{5}}$. $\boxed{y = -\frac{3}{5}(x-2) - 1}$